On the Chordality of Ordinary Differential Triangular Decomposition in Top-down Style

Chenqi Mou

BDBC–LMIB–School of Mathematical Sciences
Beihang University, China

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Polynomial system solving: triangular decomposition

$$\mathbb{R}[x_1, \ldots, x_n],\ x_1 < \cdots < x_n$$

$$\begin{array}{l}
T_1(x_1, \ldots, \boxed{x_{s_1}}) \\
T_2(x_1, \ldots, x_{s_1}, \ldots, \boxed{x_{s_2}}) \\
T_3(x_1, \ldots, x_{s_1}, \ldots, x_{s_2}, \ldots, \boxed{x_{s_3}}) \\
\vdots \\
T_r(x_1, \ldots, x_{s_1}, \ldots, x_{s_2}, \ldots, x_{s_3}, \ldots, \ldots, \boxed{x_{s_r}})
\end{array}$$

- polynomial generalization of matrices in echelon forms
- different kinds of triangular sets: characteristic [Wu 86], regular [Kalkbrener 93, Yang & Zhang 94], normal [Lazard 91, Gao & Chou 92], simple (squarefree) [Wang 98, Bächler et.al 12], etc.

Triangular decomposition in top-down style

Triangular decomposition

$$\text{Polynomial set } \mathcal{F} \subset \mathbb{R}[x_1, \ldots, x_n]$$
$$\Downarrow$$
$$\text{Triangular sets } \mathcal{T}_1, \ldots, \mathcal{T}_t \text{ s.t. } \mathsf{Z}(\mathcal{F}) = \bigcup_{i=1}^{t} \mathsf{Z}(\mathcal{T}_i / \operatorname{ini}(\mathcal{T}_i))$$

- solving $\mathcal{F} = 0 \Longrightarrow$ solving all $\mathcal{T}_i = 0$
- polynomial generalization of Gaussian elimination

Top-down style: variables handled in a decreasing order $x_n, x_{n-1}, \ldots, x_1$

- widely used strategy [Wang 1993, 1998, 2000], [Chai, Gao, Yuan 2008]
- the closest to Gaussian elimination

Matrix in echelon form $\qquad$ Triangular set

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 & x_2 & x_3 \\ * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

Gaussian elimination $\qquad$ Top-down tyle

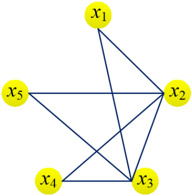
Chordal graphs

Perfect elimination ordering / chordal graph

$G = (V, E)$ a graph with $V = \{x_1, \ldots, x_n\}$:
An ordering $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ of the vertices is called a perfect elimination ordering of G if for each $j = i_1, \ldots, i_n$, the restriction of G on

$$X_j = \{x_j\} \cup \{x_k : x_k < x_j \text{ and } (x_k, x_j) \in E\}$$

is a clique. A graph G is chordal if it has a perfect elimination ordering.



Chordal graphs: no new fill-ins in Cholesky factorization $\Longrightarrow$ sparse Gaussian elimination [Parter 61, Rose 70, Gilbert 94]

Chordal graphs in triangular decomposition

Chordal graph $\Longrightarrow$ Gaussian elimination $\Longrightarrow$ Triangular decomposition in top-down style: study the behaviors and design sparse versions of triangular decomposition in top-down style

- [Cifuentes & Parrilo 2017]: chordal networks, connections between chordal graphs and triangular decomposition first established
- [Mou & Bai 2018]: chordal graph in general algorithms for triangular decomposition in top-down style, Wang's algorithm (simply structured) perserves chordality
- [Mou, Bai & Lai 2019]: regular decomposition based on subresultant regular subchains (SRS) preserves chordality, sparse triangular decomposition in top-down style
- [Mou, & Lai 2020]: simple decomposition based on SRS preserves chordality

In this talk: algebraic case $\Longrightarrow$ ordinary differential case

Ordinary differential polynomial ring

Ordinary differential polynomial ring $\mathbb{K}\{\boldsymbol{x}\}$

- $\mathbb{K}$: ordinary differential field of characteristic 0 with a derivation $\frac{\mathrm{d}}{\mathrm{d}t}$
- $x_1 < \cdots < x_n$: differential indeterminates, $x_{ij} := \frac{\mathrm{d}^j x_i}{\mathrm{d}t^j}$
- $\mathbb{K}\{\boldsymbol{x}\}$: polynomial ring over $\mathbb{K}$ in $\{x_{ij} | i = 1, \ldots, n, j \in \mathbb{Z}_{\geq 0}\}$ (infinite), equipped with $\frac{\mathrm{d}}{\mathrm{d}t}$

We fix a differential ordering $<_d$ on all the derivatives as

$$
\begin{aligned}
& x_1 <_d x_{11} <_d x_{12} <_d \cdots \\
<_d\ & x_2 <_d x_{21} <_d x_{22} <_d \cdots \\
& \vdots \\
<_d\ & x_n <_d x_{n1} <_d x_{n2} <_d \cdots .
\end{aligned}
$$

Differential triangular decomposition

For a differential polynomial $F \in \mathbb{K}\{\boldsymbol{x}\}$:

- its lead $\mathrm{ld}(F)$: greatest derivative in F w.r.t. $<_d$
- its order $\mathrm{ord}(F)$: index of greatest differential indeterminate w.r.t. $<$

Differential triangular set

A finite ordered set $\mathcal{T} = [T_1, \ldots, T_r]$ of differential polynomials in $\mathbb{K}\{\boldsymbol{x}\}$ is called a weak differential triangular set if $0 < \mathrm{ord}(T_1) < \cdots < \mathrm{ord}(T_r)$.

- Extended to a differential triangular system $[\mathcal{T}, \mathcal{U}]$: $\mathcal{T} \subset \mathbb{K}\{\boldsymbol{x}\}$ equations, $\mathcal{U} \subset \mathbb{K}\{\boldsymbol{x}\}$ inequations

Differential triangular decomposition

Differential polynomial set $\mathcal{F} \subset \mathbb{K}\{\boldsymbol{x}\}$

$$\Downarrow$$

Weak differential triangular systems $[\mathcal{T}_1, \mathcal{U}_1], \ldots, [\mathcal{T}_t, \mathcal{U}_t]$
s.t. $\mathsf{Z}(\mathcal{F}) = \bigcup_{i=1}^{t} \mathsf{Z}(\mathcal{T}_i/\mathcal{U}_i)$

Do algorithms for differential triangular decomposition in top-down style preserve the chordality?

Algebraic VS differential associated graphs

Algebraic

$\mathbb{K}[\boldsymbol{x}]$, $x_1, \ldots, x_n$ variables

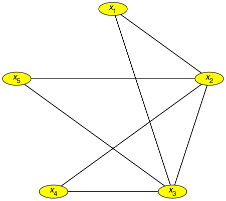
Differential

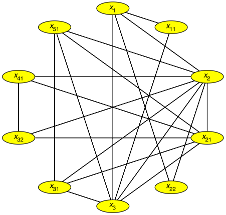
$\mathbb{K}\{\boldsymbol{x}\}$ $\begin{cases} x_1, \ldots, x_n & \text{differential indeterminates} \\ x_{ij} & \text{variables} \end{cases}$

$\mathcal{F} \subset \mathbb{K}\{\boldsymbol{x}\}$: its differential associated graph $\tilde{G}(F)$ (undirected)

- Vertices: differential indeterminates appearing in $\mathcal{F}$
- Edegs: $\{(x_i, x_j) : \exists F \in \mathcal{F}$ s.t. F contains $x_i, x_j\}$

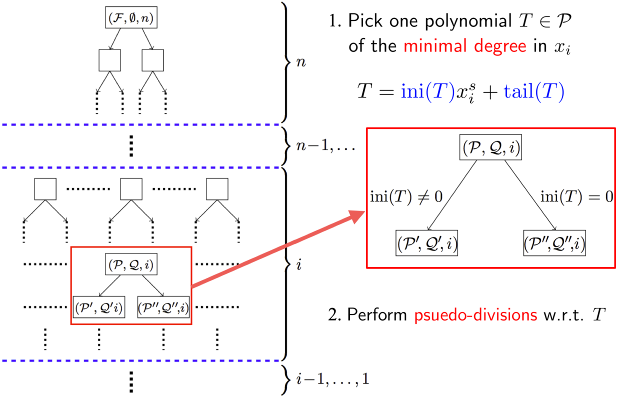
Example: $\mathbb{Q}(t)\{x_1, \ldots, x_5\}$, $\mathcal{F} = \{tx_{22} + x_2 + x_1, x_3 + tx_{11} + x_1, (1+t)x_{41}^2 + x_{21} + tx_2 + x_{32}^2, x_{51}^2 + x_{31}x_2 + x_3x_{21}\}$ **Differential** VS **Algebraic**





Wang's method for algebraic triangular decomposition

Wang's algorithm [Wang 93]: top-down style, based on psuedo-divisions, binary decomposition tree



1. Pick one polynomial $T \in \mathcal{P}$ of the minimal degree in x_i

$$T = \mathrm{ini}(T) x_i^s + \mathrm{tail}(T)$$

2. Perform psuedo-divisions w.r.t. T

Theorem: Wang's method applied to $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$, chordal

$\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

For any node $(\mathcal{P}, \mathcal{Q}, i)$ in the binary decomposition tree, $G(\mathcal{P}) \subset G(\mathcal{F})$

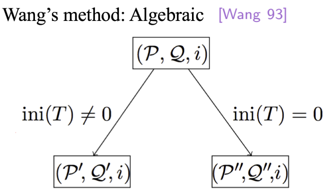
Wang's algorithm for differential triangular decomposition

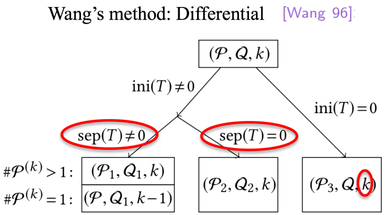
Algebraic: $G \in \mathbb{K}[\boldsymbol{x}]$

- $\mathrm{lv}(G)$: greatest variable w.r.t. $<$
- $G = \mathrm{ini}(G)\,\mathrm{lv}(G)^d + \mathrm{tail}(F)$ ($\mathrm{ini}(G)$ involves no $\mathrm{lv}(G)$)
- $\mathrm{ini}(G)^s F = QG + R$

Differential: $G \in \mathbb{K}\{\boldsymbol{x}\}$

- $\mathrm{ld}(G)$: greatest derivative w.r.t. $<_d$
- $G = \mathrm{ini}(G)\,\mathrm{ld}(G)^d + \mathrm{tail}(G)$ ($\mathrm{ord}(\mathrm{ini}(G))$ may equal $\mathrm{ord}(G)$)
- $\mathrm{sep}(G)$: formal derivative w.r.t. $\mathrm{ld}(G)$
- $\mathrm{ini}(G)^{s_1}\mathrm{sep}(G)^{s_2} F = [G] + R$





Theorem: Wang's method applied to $\mathcal{F} \subset \mathbb{K}\{\boldsymbol{x}\}$, chordal

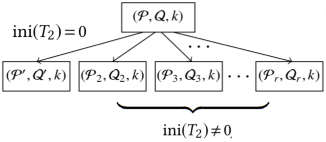
$\mathcal{F} \subset \mathbb{K}\{\boldsymbol{x}\}$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

For any node $(\mathcal{P}, \mathcal{Q}, i)$ in the decomposition tree, $\tilde{G}(\mathcal{P}) \subset \tilde{G}(\mathcal{F})$

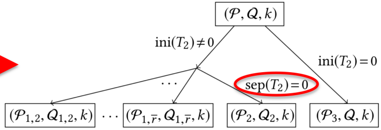
SRS-based algorithm for differential triangular decomposition

- SRS is used to find the pseudo-gcd of two differential polynomials: $H_2, \dots, H_r := \mathrm{SubRegSubchain}(T_1, T_2, x)$
- The differential case is almost the same as the algebraic one: only the variable now is the derivative x_{ij}

Algebraic case [Wang 00]



Differential case [Hubert 04]



Theorem: SRS-based algorithm applied to $\mathcal{F} \subset \mathbb{K}\{\boldsymbol{x}\}$, chordal

$\mathcal{F} \subset \mathbb{K}\{\boldsymbol{x}\}$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

For any node $(\mathcal{P}, \mathcal{Q}, i)$ in the decomposition tree, $\tilde{G}(\mathcal{P}) \subset \tilde{G}(\mathcal{F})$

Concluding remarks

- Propose the concept of differential chordal graphs
- Prove that two typical algorithms for ordinary differential triangular decomposition preserve chordality $\Longrightarrow$ sparse algorithms as in the algebraic case? [Mou, Bai & Lai 19]
- What about the partial differential case? HARD
  - Definition of partial differential triangular sets: based on leads, $\frac{\partial x_1}{\partial u}$ and $\frac{\partial x_1}{\partial v}$
  - Property of coherence: Δ-polynomials, against the top-down strategy

Thanks for watching!