On the chordality of polynomial sets in triangular decomposition in top-down style

Chenqi Mou

joint work with **Yang Bai**

LMIB–School of Mathematics and Systems Science

Beihang University, China

Computer Mathematics Seminar

April 26, 2018

Chordal graph

Perfect elimination ordering / chordal graph

$G = (V, E)$ a graph with $V = \{x_1, \ldots, x_n\}$:
An ordering $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ of the vertexes is called a *perfect elimination ordering* of G if for each $j = i_1, \ldots, i_n$, the restriction of G on

$$X_j = \{x_j\} \cup \{x_k : x_k < x_j \text{ and } (x_k, x_j) \in E\}$$

is a clique. A graph G is said to be *chordal* if there exists a perfect elimination ordering of it.

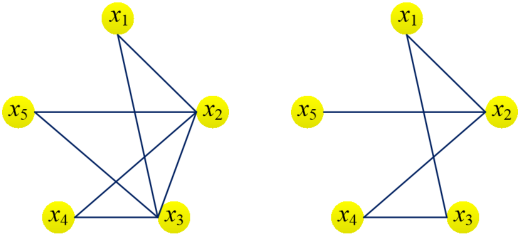


Figure: Chordal VS non-chordal graphs

Chordal graph

Equivalent conditions

$G = (V, E)$ *chordal* $\iff$ for any cycle C contained in G of four or more vertexes, there is an edge $e \in E \setminus C$ connects two vertexes in C.

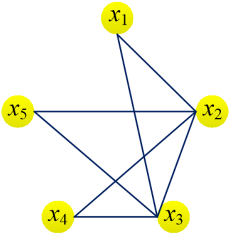


Figure: An illustrative chordal graph

- A chordal graph is also called a triangulated one.

Triangular set and decomposition

Triangular set in $\mathbb{K}[x_1, \ldots, x_n]$ with $x_1 < \cdots < x_n$

$$T_1(x_1, \ldots, x_{s_1})$$
$$T_2(x_1, \ldots, x_{s_1}, \ldots, x_{s_2})$$
$$T_3(x_1, \ldots, x_{s_1}, \ldots, x_{s_2}, \ldots, x_{s_3})$$
$$\vdots$$
$$T_r(x_1, \ldots, x_{s_1}, \ldots, x_{s_2}, \ldots, x_{s_3}, \ldots, \ldots, x_{s_r})$$

Triangular decomposition

Polynomial sets $\mathcal{F} \subset \mathbb{K}[x_1, \ldots, x_n]$

$$\Downarrow$$

Triangular sets $\mathcal{T}_1, \ldots, \mathcal{T}_t$ s.t. $\mathsf{Z}(\mathcal{F}) = \bigcup_{i=1}^{t} \mathsf{Z}(\mathcal{T}_i / \operatorname{ini}(\mathcal{T}_i))$

$\rightsquigarrow$ Solving $\mathcal{F} = 0 \Longrightarrow$ solving all $\mathcal{T}_i = 0$

$\rightsquigarrow$ Multivariate generalization of Gaussian elimination

Inspired by the pioneering works of

D. Cifuentes P.A. Parrilo (from MIT)

on triangular sets and chordal graphs

[Cifuentes and Parrilo 2017]: chordal networks of polynomial systems

- Connections between triangular sets and chordal graphs
- Algorithms for computing triangular sets due to Wang become more efficient when the input polynomial set is chordal ($\Longrightarrow$ Why?)

$\rightsquigarrow$ [Cifuents and Parrilo 2016]: Gröbner bases and chordal graphs

Chordal networks

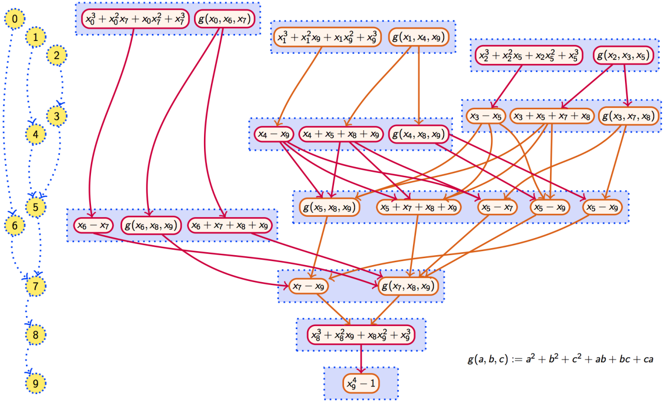


Figure: A chordal network (borrowed from Parrilo's slides)

- Elimination tree / triangular decomposition clique-wisely

Associated graphs of polynomial sets

$F \in \mathbb{K}[x_1, \ldots, x_n]$ a polynomial: the (variable) *support* of F, $\mathrm{supp}(F)$, is the set of variables in $x_1, \ldots, x_n$ which effectively appear in F

- $\mathrm{supp}(\mathcal{F}) := \cup_{F \in \mathcal{F}} \mathrm{supp}(F)$ for $\mathcal{F} \subset \mathbb{K}[x_1, \ldots, x_n]$

Associated graphs

$\mathcal{F} \subset \mathbb{K}[x_1, \ldots, x_n]$, *associated graph* $G(\mathcal{F})$ of $\mathcal{F}$ is an undirected graph:

(a) vertexes of $G(\mathcal{F})$: the variables in $\mathrm{supp}(\mathcal{F})$

(b) edge (x_i, x_j) in $G(\mathcal{F})$: if there exists one polynomial $F \in \mathcal{F}$ with $x_i, x_j \in \mathrm{supp}(F)$

Chordal polynomial set

A polynomial set $\mathcal{F} \subset \mathbb{K}[x_1, \ldots, x_n]$ is said to be *chordal* if $G(\mathcal{F})$ is chordal.

Associated graphs of polynomial sets

$$\mathbb{K}[x_1, \ldots, x_5]$$

$$\mathcal{P} = \{x_2 + x_1, x_3 + x_1, x_4^2 + x_2, x_4^3 + x_3, x_5 + x_2, x_5 + x_3 + x_2\}$$

$$\mathcal{Q} = \{x_2 + x_1, x_3 + x_1, x_3, x_4^2 + x_2, x_4^3 + x_3, x_5 + x_2\}$$

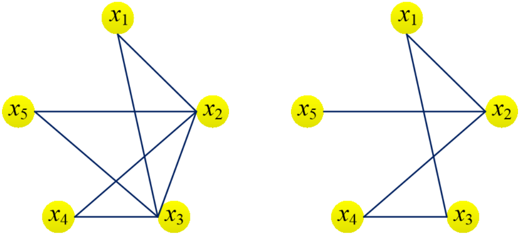
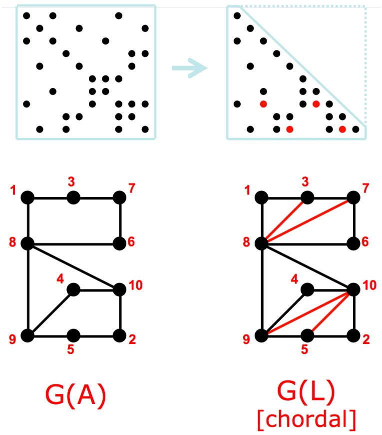


Figure: Associated graphs $G(\mathcal{P})$ (chordal) and $G(\mathcal{Q})$ (not chordal)

Chordal graphs in Gaussian elimination

New fill-ins in Cholesky factorization of a matrix $A = LL^t$ (credits to J. Gilbert)



Matrices with chordal graphs: no new fill-ins (subgraphs) $\Longrightarrow$ sparse Gaussian elimination [Parter 61, Rose 70, Gilbert 94]

Triangular decomposition in top-down style

The variables are handled in a strictly decreasing order: $x_n, x_{n-1}, \ldots, x_1$

- widely used strategy [Wang 1993, 1998, 2000], [Chai, Gao, Yuan 2008]
- the closest to Gaussian elimination
- algorithms due to Wang are mostly in top-down style (!!)

Matrix in echelon form — Triangular set

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 & x_2 & x_3 \\ * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

Gaussian elimination — Top-down triangular decomposition

Problems

1. Chordal graphs in Gaussian elimination $\Longrightarrow$ Chordal graphs in triangular decomposition in top-down style: multivariate generalization
   - Changes of graph structures of the polynomials in triangular decomposition
   - relationships (like inclusion) between associated graphs of computed triangular sets and the input polynomial set
2. Sparse Gaussian elimination $\Longrightarrow$ sparse triangular decomposition in top-down style: multivariate generalization, on-going work
   - sparse Gröbner bases [Faugère, Spaenlehauer, Svartz 2014]
   - sparse FGLM algorithms [Faugère, Mou 2011, 2017]

Reduction to a triangular set from a chordal polynomial set

$\mathcal{P} \subset \mathbb{K}[x_1, \ldots, x_n]$: $\mathcal{P}^{(i)} = \{P \in \mathcal{P} : \mathrm{lv}(P) = x_i\}$

Proposition

$\mathcal{P} \subset \mathbb{K}[x_1, \ldots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

Let $T_i \in \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial with $\mathrm{lv}(T_i) = x_i$ and $\mathrm{supp}(T_i) \subset \mathrm{supp}(\mathcal{P}^{(i)})$. Then $\mathcal{T} = [T_1, \ldots, T_n]$ is a triangular set, and $G(\mathcal{T}) \subset G(\mathcal{P})$.

$\rightsquigarrow$In particular, $\mathrm{supp}(T_i) = \mathrm{supp}(\mathcal{P}^{(i)}) \Longrightarrow G(\mathcal{T}) = G(\mathcal{P})$

$$\begin{array}{cccccc} \mathcal{P} = \{\mathcal{P}^{(1)}, & \mathcal{P}^{(2)}, & \ldots, & \mathcal{P}^{(n)}\} : & & G(\mathcal{P}) \\ \Downarrow & \Downarrow & & \Downarrow & & \cup \\ \mathcal{T} = [\ T_1, & T_2, & \ldots, & T_n\ ] : & & G(\mathcal{T}) \end{array}$$

An counter-example for non-chordal polynomial sets

This proposition does not necessarily hold in general if the polynomial set $\mathcal{P}$ is not chordal.

$$\mathcal{Q} = \{x_2 + x_1, x_3 + x_1, x_3, x_4^2 + x_2, x_4^3 + x_3, x_5 + x_2\}$$

$$\Downarrow$$

$$\mathcal{T} = [x_2 + x_1, x_3 + x_1, -x_2x_4 + x_3, x_5 + x_2]$$

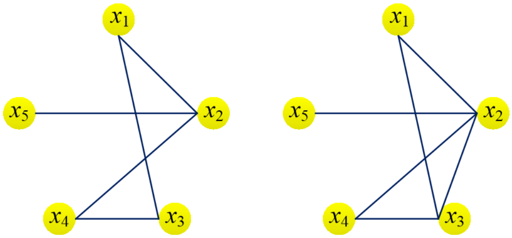


Figure: The associated graphs $G(\mathcal{Q})$ and $G(\mathcal{T})$

Reduction w.r.t. one variable in triangular decomposition

Theorem

$\mathcal{P} \subset \mathbb{K}[x_1, \ldots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

Let $T \in \mathbb{K}[x_1, \ldots, x_n]$ with $\mathrm{lv}(T) = x_n$ and $\mathrm{supp}(T) \subset \mathrm{supp}(\mathcal{P}^{(n)})$, and $\mathcal{R} \subset \mathbb{K}[x_1, \ldots, x_n]$ such that $\mathrm{supp}(\mathcal{R}) \subset \mathrm{supp}(\mathcal{P}^{(n)}) \setminus \{x_n\}$. Then for the polynomial set

$$\tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}^{(1)}, \ldots, \tilde{\mathcal{P}}^{(n-1)}, T\},$$

where $\tilde{\mathcal{P}}^{(k)} = \mathcal{P}^{(k)} \cup \mathcal{R}^{(k)}$ for $k = 1, \ldots, n-1$, we have $G(\tilde{\mathcal{P}}) \subset G(\mathcal{P})$

$\rightsquigarrow$In particular, $\mathrm{supp}(T) = \mathrm{supp}(\mathcal{P}^{(n)}) \Longrightarrow G(\tilde{\mathcal{P}}) = G(\mathcal{P})$

- commonly-used reduction in top-down triangular decomposition

$$\begin{array}{cccccc}
\mathcal{P} = \{\mathcal{P}^{(1)}, & \mathcal{P}^{(2)}, & \ldots, & \mathcal{P}^{(n)}\}: & & G(\mathcal{P}) \\
\Downarrow & \Downarrow & & \Downarrow & & \cup \\
\tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}^{(1)}, & \tilde{\mathcal{P}}^{(2)}, & \ldots, & T \}: & & G(\tilde{\mathcal{P}}) \\
/\!/ & \| & & s.t. & & \\
\mathcal{P}^{(1)} \cup \mathcal{R}^{(1)} & \mathcal{P}^{(2)} \cup \mathcal{R}^{(2)} & & \mathrm{supp}(T) \subset \mathrm{supp}(\mathcal{P}^{(n)}) & &
\end{array}$$

Some notations

mapping f_i

$$f_i : 2^{\mathbb{K}[\boldsymbol{x}_i]\setminus\mathbb{K}[\boldsymbol{x}_{i-1}]} \to (\mathbb{K}[\boldsymbol{x}_i] \setminus \mathbb{K}[\boldsymbol{x}_{i-1}]) \times 2^{\mathbb{K}[\boldsymbol{x}_{i-1}]}$$

$$\mathcal{P} \mapsto (T, \mathcal{R})$$

s.t $\mathrm{supp}(T) \subset \mathrm{supp}(\mathcal{P})$ and $\mathrm{supp}(\mathcal{R}) \subset \mathrm{supp}(\mathcal{P})$ (where $\mathbb{K}[\boldsymbol{x}_0] = \mathbb{K}$).

$\mathcal{P} \subset \mathbb{K}[x_1, \ldots, x_n]$ and a fixed integer i $(1 \leq i \leq n)$, suppose that $(T_i, \mathcal{R}_i) = f_i(\mathcal{P}^{(i)})$ for some f_i. For $j = 1, \ldots, n$, define

$$\mathrm{red}_i(\mathcal{P}^{(j)}) := \begin{cases} \mathcal{P}^{(j)}, & \text{if } j > i \\ \{T_i\}, & \text{if } j = i \\ \mathcal{P}^{(j)} \cup \mathcal{R}_i^{(j)}, & \text{if } j < i \end{cases}$$

and $\mathrm{red}_i(\mathcal{P}) := \cup_{j=1}^{n} \mathrm{red}_i(\mathcal{P}^{(j)})$. In particular, write

$$\boxed{\overline{\mathrm{red}}_i(\mathcal{P}) := \mathrm{red}_i(\mathrm{red}_{i+1}(\cdots(\mathrm{red}_n(\mathcal{P}))\cdots))}$$

The above theorem becomes

$G(\mathrm{red}_n(\mathcal{P})) \subset G(\mathcal{P})$, and the equality holds if $\mathrm{supp}(T_n) = \mathrm{supp}(\mathcal{P}^{(n)})$.

Reduction w.r.t. all variables in triangular decomposition

$$\begin{array}{rcccccl}
\mathcal{P} = \{\mathcal{P}^{(1)}, & \mathcal{P}^{(2)}, & \ldots, & \mathcal{P}^{(n-1)}, & \mathcal{P}^{(n)}\}: & & G(\mathcal{P}) \\
\Downarrow & \Downarrow & & \Downarrow & \Downarrow & & \cup \\
\mathrm{red}_n(\mathcal{P}) = \{\tilde{\mathcal{P}}^{(1)}, & \tilde{\mathcal{P}}^{(2)}, & \ldots, & \tilde{\mathcal{P}}^{(n-1)}, & T_n\}: & & G(\mathrm{red}_n(\mathcal{P})) \\
\Downarrow & \Downarrow & & \Downarrow & \Downarrow & & \boxed{??} \\
\overline{\mathrm{red}}_{n-1}(\mathcal{P}) = \{\tilde{\tilde{\mathcal{P}}}^{(1)}, & \tilde{\tilde{\mathcal{P}}}^{(2)}, & \ldots, & T_{n-1}, & T_n\}: & & G(\overline{\mathrm{red}}_{n-1}(\mathcal{P})) \\
 & & \vdots & & & & \boxed{??} \\
\overline{\mathrm{red}}_1(\mathcal{P}) = \{T_1, & T_2, & \ldots, & T_{n-1}, & T_n\}: & & G(\overline{\mathrm{red}}_1(\mathcal{P}))
\end{array}$$

Proposition

$\mathcal{P} \subset \mathbb{K}[x_1, \ldots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

For each i $(1 \leq i \leq n)$, suppose that $(T_i, \mathcal{R}_i) = f_i(\overline{\mathrm{red}}_{i+1}(\mathcal{P})^{(i)})$ for some f_i and $\mathrm{supp}(T_i) = \mathrm{supp}(\overline{\mathrm{red}}_{i+1}(\mathcal{P})^{(i)})$. Then

$$G(\overline{\mathrm{red}}_1(\mathcal{P})) = \cdots = G(\overline{\mathrm{red}}_{n-1}(\mathcal{P})) = G(\mathrm{red}_n(\mathcal{P})) = G(\mathcal{P}).$$

Counter example for successive inclusions

$\operatorname{supp}(T_i) \subset \operatorname{supp}(\overline{\operatorname{red}}_{i+1}(\mathcal{P})^{(i)})$: then in general we will NOT have

$$G(\overline{\operatorname{red}}_1(\mathcal{P})) \subset \cdots \subset G(\overline{\operatorname{red}}_{n-1}(\mathcal{P})) \subset G(\operatorname{red}_n(\mathcal{P})) \subset G(\mathcal{P})$$

Example

$$\mathcal{P} = \{x_2 + x_1, x_3 + x_1, x_4^2 + x_2, x_4^3 + x_3, x_5 + x_2, x_5 + x_3 + x_2\}$$

$$\mathcal{Q} = \operatorname{red}_5(\mathcal{P}) = \{x_2 + x_1, x_3 + x_1, x_3, x_4^2 + x_2, x_4^3 + x_3, x_5 + x_2\}$$

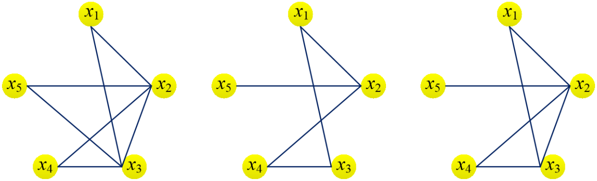
$$\Downarrow$$

$$T_4 = \operatorname{prem}(x_4^3 + x_3, x_4^2 + x_2) = -x_2x_4 + x_3,$$

$$\mathcal{R}_4 = \{\operatorname{prem}(x_4^2 + x_2, -x_2x_4 + x_3)\} = \{x_3^2 - x_2^3\},$$

$$\Downarrow$$

$$\mathcal{Q}' := \overline{\operatorname{red}}_4(\mathcal{P}) = \{x_2 + x_1, x_3 + x_1, x_3^2 - x_2^3, x_3, -x_2x_4 + x_3, x_5 + x_2\}.$$



Subgraphs of the input chordal graph

Theorem

$\mathcal{P} \subset \mathbb{K}[x_1, \ldots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

For each $i = n, \ldots, 1$, $\boxed{G(\overline{\mathrm{red}}_i(\mathcal{P})) \subset G(\mathcal{P})}$.

Corollary

$\mathcal{P} \subset \mathbb{K}[x_1, \ldots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

If $\mathcal{T} := \overline{\mathrm{red}}_1(\mathcal{P})$ does not contain any nonzero constant, then $\mathcal{T}$ forms a triangular set such that $G(\mathcal{T}) \subset G(\mathcal{P})$.

- $\mathcal{T}$ above: the main component in the triangular decomposition
- Valid for ANY algorithms for triangular decomposition in top-down style
- Problem: what about the other triangular sets?

Wang's method: algorithm

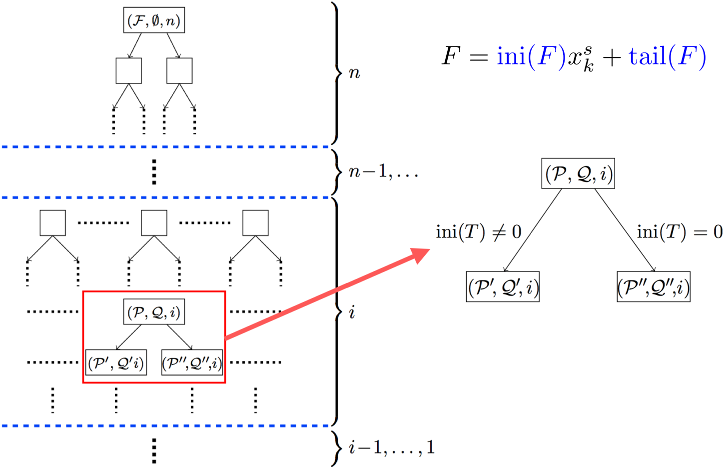
[Wang 93]: Wang's method, simply-structured algorithm for triangular decomposition in top-down style

Algorithm 1: Wang's method for triangular decomposition $\Psi := \mathrm{TriDecWang}(\mathcal{P})$

Input: $\mathcal{F}$, a polynomial set in $\mathbb{K}[\boldsymbol{x}]$
Output: Ψ, a set of finitely many triangular systems which form a triangular decomposition of $\mathcal{F}$

$\Phi := \{(\mathcal{F}, \emptyset, n)\}$;
while $\Phi \neq \emptyset$ **do**
  $(\mathcal{P}, \mathcal{Q}, i) :=$ pop(Φ);
  if $i = 0$ **then**
    $\Psi := \Psi \cup \{(\mathcal{P}, \mathcal{Q})\}$;
    Break;
  while $\#(\mathcal{P}^{(i)}) > 1$ **do**
    $T :=$ a polynomial in $\mathcal{P}^{(i)}$ with minimal degree in x_i;
    $\Phi := \Phi \cup \{(\mathcal{P} \setminus \{T\} \cup \{\mathrm{ini}(T), \mathrm{tail}(T)\}, \mathcal{Q}, i)\}$;
    $\overline{\mathcal{P}} := \mathcal{P}^{(i)} \setminus \{T\}$;
    $\mathcal{P} := \mathcal{P} \setminus \overline{\mathcal{P}}$;
    for $P \in \mathcal{P}^{(i)}$ **do**
      $\mathcal{P} := \mathcal{P} \cup \{\mathrm{prem}(P, T)\}$;
    $\mathcal{Q} := \mathcal{Q} \cup \{\mathrm{ini}(T)\}$;
  $\Phi := \Phi \cup \{(\mathcal{P}, \mathcal{Q}, i-1)\}$;
for $(\mathcal{P}, \mathcal{Q}) \in \Psi$ **do**
  if $\mathcal{P}$ *contains a non-zero constant* **then**
    $\Psi := \Psi \setminus \{(\mathcal{P}, \mathcal{Q})\}$
return Ψ

Wang's method: binary decomposition tree



$$\mathcal{P}' := \mathcal{P} \setminus \mathcal{P}^{(i)} \cup \{T\} \cup \{\text{prem}(P, T) : P \in \mathcal{P}\}, \quad \mathcal{Q}' := \mathcal{Q} \cup \{\text{ini}(T)\},$$

$$\mathcal{P}'' := \mathcal{P} \setminus \{T\} \cup \{\text{ini}(T), \text{tail}(T)\}, \quad \mathcal{Q}'' := \mathcal{Q},$$

Wang's method: left child

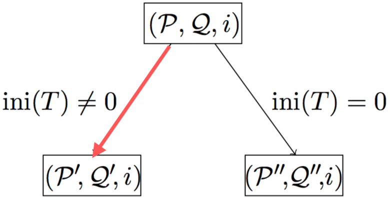
Proposition: Wang's method applied to $\mathcal{F} \subset \mathbb{K}[x_1, \ldots, x_n]$, chordal

$\mathcal{F} \subset \mathbb{K}[x_1, \ldots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

$(\mathcal{P}, \mathcal{Q}, i)$ arbitrary node in the binary decomposition tree such that $G(\mathcal{P}) \subset G(\mathcal{F})$, $T \in \mathcal{P}$ with minimal degree in x_i. Denote

$$\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}^{(i)} \cup \{T\} \cup \{\mathrm{prem}(P, T): \ P \in \mathcal{P}^{(i)}\}.$$

Then $G(\mathcal{P}') \subset G(\mathcal{F})$.



$G(\mathcal{P}') \subset G(\mathcal{F})$ on the conditions that $G(\mathcal{F})$ is chordal and $G(\mathcal{P}) \subset G(\mathcal{F})$

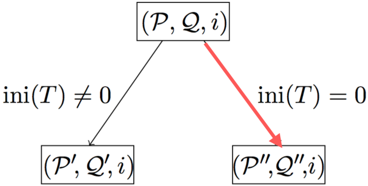
Wang's method: right child

Proposition

$(\mathcal{P}, \mathcal{Q}, i)$ arbitrary node in the binary decomposition tree, $T \in \mathcal{P}^{(i)}$ with minimal degree in x_i. Denote

$$\mathcal{P}'' = \mathcal{P} \setminus \{T\} \cup \{\mathrm{ini}(T), \mathrm{tail}(T)\}.$$

Then $G(\mathcal{P}'') \subset G(P)$.
$\rightsquigarrow$In particular, $\mathrm{supp}(\mathrm{tail}(T)) = \mathrm{supp}(T) \Longrightarrow G(\mathcal{P}'') = G(\mathcal{P})$.



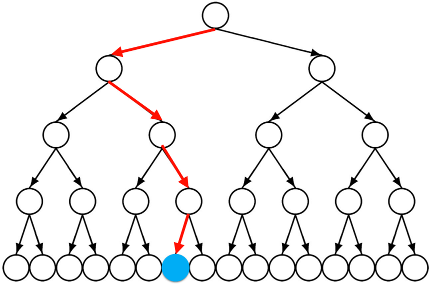
$G(\mathcal{P}'') \subset G(P)$ under no conditions

Wang's method: any node

Theorem: Wang's method applied to $\mathcal{F} \subset \mathbb{K}[x_1, \dots, x_n]$, chordal

$\mathcal{F} \subset \mathbb{K}[x_1, \dots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

For any node $(\mathcal{P}, \mathcal{Q}, i)$ in the binary decomposition tree, $G(\mathcal{P}) \subset G(\mathcal{F})$



Corollary: Wang's method applied to $\mathcal{F} \subset \mathbb{K}[x_1, \dots, x_n]$, chordal

$\mathcal{F} \subset \mathbb{K}[x_1, \dots, x_n]$ chordal, $x_1 < \cdots < x_n$ perfect elimination ordering:

For any triangular set $\mathcal{T}$ computed by Wang's method, $G(\mathcal{T}) \subset G(\mathcal{F})$

Complexity analysis for triangular decomposition in top-down style

Chordal completion

For a graph G, another graph G' is called a *chordal completion* of $\mathcal{G}$ if G' is chordal with G as its subgraph.

The *treewidth* of a graph G is defined to be the minimum of the sizes of the largest cliques in all the possible chordal completions of G.

- many NP-complete problems related to graphs can be solved efficiently for graphs of bounded treewidth [Arnborg, Proskurowski 1989]
- Complexities for computing Gröbner bases for polynomial sets with small treewidth [Cifuents and Parrilo 2016]

Reminding you of the inclusion of graphs for Wang's method

The input chordal associated graph: upper bound

- Complexities for triangular decomposition: first for polynomial sets with chordal graphs / small treewidth

Variable sparsity of polynomial sets

Variable sparsity

$G(\mathcal{F}) = (V, E)$ associated graph of $\mathcal{F} = \{F_1, \ldots, F_r\} \subset \mathbb{K}[x_1, \ldots, x_n]$. Define the *variable sparsity* $s_v(\mathcal{F})$ of $\mathcal{F}$ as

$$s_v(\mathcal{F}) = |E| / \binom{2}{|V|},$$

denominator: edge number of a complete graph of $|V|$ vertexes

$G(\mathcal{F})$ can be extended to a weighted graph $G^w(\mathcal{F})$ by associating the number $\#\{F \in \mathcal{F} : x_i, x_j \in \mathrm{supp}(F)\}$ to each edge (x_i, x_j) of $G(\mathcal{F})$

Weighted variable sparsity

the *weighted variable sparsity* $s_v^w(\mathcal{F})$ of $\mathcal{F}$ can be defined as

$$s_v^w(\mathcal{F}) = \frac{\sum_{e \in E} w_e}{r \cdot \binom{2}{|V|}},$$

where r is the number of polynomials in $\mathcal{F}$.

Sparse triangular decomposition

A refined algorithm for regular decomposition

Input: a polynomial set $\mathcal{F} \subset \mathbb{K}[\boldsymbol{x}]$
Output: a variable ordering $\overline{\boldsymbol{x}}$ and a regular decomposition Φ of $\mathcal{F}$ with respect to $\overline{\boldsymbol{x}}$

1. Compute the variable sparsity s_v of $\mathcal{F}$
2. If s_v is smaller than some sparsity threshold s_0 ($\mathcal{F}$ is sparse), then
    1. If $G(\mathcal{F})$ is chordal, then compute its perfect elimination ordering $\overline{\boldsymbol{x}}$ [1]
    2. Else compute its chordal completion $\overline{G}(\mathcal{F})$ [2] and a perfect elimination ordering $\overline{\boldsymbol{x}}$ of $\overline{G}(\mathcal{F})$
3. Compute the regular decomposition of $\mathcal{F}$ with respect to $\overline{\boldsymbol{x}}$ with a top-down algorithm [3]

[1] [Rose, Tarjan, and Lueker 1976]
[2] [Bodlaender and Koster 2008]
[3] Say, [Wang 2000]

Sparse triangular decomposition

A sparse polynomial system arising from the lattice reachability problem [Cifuentes and Parrilo 2017], [Diaconis, Eisenbud, Sturmfels 1998]

$$\mathcal{F}_i := \{x_k x_{k+3} - x_{k+1} x_{k+2} : k = 1, 2, \ldots, i \}, \quad i \in \mathbb{Z}_{>0}$$

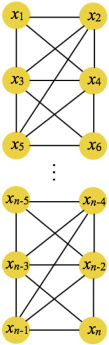


Figure: Associated graph of $\mathcal{F}_i$

Sparse triangular decomposition

Comparisons of timings for computing regular decomposition of one class of chordal and variable sparse polynomials [Cifuentes and Parrilo 2017]

$$\mathcal{F}_i := \{x_k x_{k+3} - x_{k+1} x_{k+2} : k = 1, 2, \ldots, i\}, \quad i \in \mathbb{Z}_{>0}$$

Table: Regular decomposition with RegSer in Epsilon: top-down

n	s_v	t_p	t_r					$\bar{t}_r$	$\bar{t}_r/t_p$
10	0.53	0.19	0.14	0.21	0.22	0.11	0.21	0.18	0.95
20	0.28	1.44	4.24	4.45	3.15	4.41	4.65	4.18	2.90
25	0.23	4.25	50.62	20.29	15.55	25.01	35.10	29.31	6.90
30	0.19	11.94	177.37	185.94	130.54	142.97	103.42	148.05	12.40
35	0.17	42.33	560.56	291.35	633.43	320.98	938.45	548.95	12.97
40	0.15	161.11	1883.64	3618.04	4289.13	4013.99	2996.37	3360.23	20.86

Table: Regular decomposition with RegularChains in Maple: not top-down

n	s_v	t_p	t_r					$\bar{t}_r$	$\bar{t}_r/t_p$
15	0.37	45.90	17.29	21.41	13.62	32.50	19.63	20.89	0.46
17	0.33	216.69	87.29	197.35	104.86	68.28	130.83	117.72	0.54
19	0.30	1303.08	415.90	308.37	780.75	221.75	831.15	511.58	0.39
21	0.27	8787.32	1823.29	2064.55	2431.49	1926.02	1593.36	1967.74	0.22

Sparse triangular decomposition

Comparisons of timings for computing regular decomposition of one class of chordal and variable sparse polynomials [Cifuentes and Parrilo 2017]

$$\mathcal{F}_i := \{x_k x_{k+3} - x_{k+1} x_{k+2} : k = 1, 2, \ldots, i\}, \quad i \in \mathbb{Z}_{>0}$$

Table: Regular decomposition with RegSer in Epsilon: top-down

n	s_v	t_p	t_r					$\bar{t}_r$	$\bar{t}_r/t_p$
10	0.53	0.19	0.14	0.21	0.22	0.11	0.21	0.18	0.95
20	0.28	1.44	4.24	4.45	3.15	4.41	4.65	4.18	2.90
25	0.23	4.25	50.62	20.29	15.55	25.01	35.10	29.31	6.90
30	0.19	11.94	177.37	185.94	130.54	142.97	103.42	148.05	12.40
35	0.17	42.33	560.56	291.35	633.43	320.98	938.45	548.95	12.97
40	0.15	161.11	1883.64	3618.04	4289.13	4013.99	2996.37	3360.23	20.86

Table: Regular decomposition with RegularChains in Maple: not top-down

n	s_v	t_p	t_r					$\bar{t}_r$	$\bar{t}_r/t_p$
15	0.37	45.90	17.29	21.41	13.62	32.50	19.63	20.89	0.46
17	0.33	216.69	87.29	197.35	104.86	68.28	130.83	117.72	0.54
19	0.30	1303.08	415.90	308.37	780.75	221.75	831.15	511.58	0.39
21	0.27	8787.32	1823.29	2064.55	2431.49	1926.02	1593.36	1967.74	0.22

Future works

- Chordality in regular decomposition in top-down style: the most popular triangular decomposition
- More other graph structures to study?

Thanks!

Recruitment

We need Postdocs!

Postdoctoral Positions in

"Scientific Data and Computing Intelligence" Research Team
led by Prof. Dongming WANG in
Beijing Advanced Innovation Center for Big Data and Brain Computing

- Basic annual payment between 250k-350k CNY
- Send your CV and brief future research plans to me (Dr.Chenqi MOU, chenqi.mou@buaa.edu.cn)
- More information: http://cmou.net/files/bdbc-postdoc-chn.pdf